

Relativistic Quantum Games in Noninertial Frames

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Abstract

We study the influence of Unruh effect on quantum non-zero sum games. In particular, we investigate the quantum Prisoners' Dilemma both for entangled and unentangled initial states and show that the acceleration of the noninertial frames disturbs the symmetry of the game. It is shown that for maximally entangled initial state, the classical strategy \hat{C} (cooperation) becomes the dominant strategy. Our investigation shows that any quantum strategy does no better for any player against the classical strategies. The miracle move of Eisert et al [2] is no more a superior move. We show that the dilemma like situation is resolved in favor of one player or the other.

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Keywords: Quantum games; Unruh effect; Noninertial frames

1 Introduction

Quantum game theory, began from the seminal paper of Meyer [1]. It deals with classical games in the domain of quantum mechanics. For the last few years much valuable work has been done in this area. Various quantum protocols have been developed and many classical games have been extended to the domain of quantum mechanics. It has been shown that quantum superposition and prior quantum entanglement between the players' states ensure quantum players to outperform the classical counterparts through quantum mechanical strategies[2-9]. Quantum entanglement is one of the powerful tools of quantum mechanics and plays the role of a kernel in quantum information and quantum computation. A prior quantum entanglement between two spatially separated parties increases the number of classical information communicated between them to twice the number of classical bits communicated in the case of unentangled state [10, 11]. Recently, the behavior of prior entanglement shared

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between two spatially separated parties has been extended to the relativistic setup in noninertial frames [12, 13, 14, 15, 16, 17] and interesting results have been obtained. Alsing *et al.* [12] have shown that the entanglement between the two modes of a free Dirac field is degraded by the Unruh effect and asymptotically reaches a nonvanishing minimum value in the limit of infinite acceleration.

In this paper, we study the influence of Unruh effect on the payoffs function of the players in the quantum non-zero sum games. In particular, we concentrate on the quantum Prisoners' Dilemma [2]. We show that the payoffs function of the players are strongly influenced by the acceleration of the noninertial frame and the symmetry of the game is disturbed. It is shown that under some particular situations, the classical strategy \hat{C} becomes the dominant strategy and the classical strategy profiles (\hat{C}, \hat{C}) and (\hat{D}, \hat{D}) are no more the Pareto optimal and the Nash equilibrium, respectively. We show that the dominance of the quantum player ceases in the presence of acceleration of the noninertial frame. In the infinite limit of acceleration, new Nash equilibrium arises. Furthermore, the dilemma like situation under every condition, we consider here, is resolved in the favor of one player or the other or both.

2 The Prisoners' Dilemma

The Prisoners' Dilemma is a well known non-zero sum game, which has a widespread applications in many areas of science. Each one of the two players (Alice and Bob) has to choose one of the two pure strategies simultaneously. The two pure strategies are called cooperation (C) and defection (D). The reward to the action of a player depends not only on his own strategy but also on the strategy of his opponent. The classical payoff matrix of the game has the structure given in Table 1. The first number in each pair of the matrix corresponds to Alice's payoff and the second number in a pair to Bob's payoff. This is a symmetric noncooperative game where each player tries to maximize his/her own payoff. The catch of the dilemma is that D is the dominant strategy, that is, rational reasoning forces each player to defect, and thereby doing substantially worse than if they would both decide to cooperate. The quantum form of the Prisoners' Dilemma was studied for the first time by Eisert et al [2].

Table 1: Payoff matrix for the classical Prisoners' Dilemma. The first entry in a pair of numbers denotes the payoff of Alice and the second entry represents Bob's payoff.

| | Bob: C | Bob: D |
|------------|----------|----------|
| Alice: C | 3, 3 | 0, 5 |
| Alice: D | 5, 0 | 1, 1 |

3 Calculation

We consider that Alice and Bob share an entangled initial state $|\psi_i\rangle = \hat{J}|00\rangle_{A,B}$ of two qubits (one for each player) at a point in flat Minkowski spacetime. The subscripts A, B of the ket stand, respectively, for Alice and Bob, which means that the first entry in the ket corresponds to Alice and the second entry corresponds to Bob. The unitary operator \hat{J} is an entangling operator and is given by

$$\hat{J} = \exp[i\frac{\gamma}{2}\hat{D}_1 \otimes \hat{D}_1], \quad (1)$$

where $\gamma \in [0, \pi/2]$ and is a measure of the degree of entanglement in the initial state. The initial state is maximally entangled when $\gamma = \pi/2$. The operator \hat{D}_1 is given by

$$\hat{D}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (2)$$

The entangling operator \hat{J} must be symmetric with respect to the interchange of the two players in order to execute a fair game and must be known to both players for the knowledge of the degree of entanglement in the initial state. The initial state, after the entangling operator is applied, becomes

$$|\psi_i\rangle = \cos\frac{\gamma}{2}|00\rangle_{A,B} + i\sin\frac{\gamma}{2}|11\rangle_{A,B}. \quad (3)$$

We consider that Bob then moves with a uniform acceleration and Alice stays stationary. Each player is equipped with a device which is sensitive only to a single mode in their respective regions. To cover Minkowski space, two different sets of Rindler coordinates (τ, ξ) (see Fig. (1)) that differ from each other by an overall change in sign and define two Rindler regions (I, II) are necessary (for detail see [12] and references therein). A uniformly accelerated particle (observer) in one Rindler region is causally disconnected from the other Rindler region at the opposite side. Thus an observer in region I has no access to the information that leaks into region II . The opposite is true for an observer in region II . An observer in region II is called anti-observer (anti-particle) of the observer in region I . The inaccessible information that leaks into the opposite region is as the system is decohered. The decoherence effects in quantum games in inertial frames are studied by a number of authors [18, 19, 20]. Particularly, in Ref. [18] the decoherence effects on quantum Prisoners' Dilemma has been studied using various quantum channels. However, the results of our calculations in the relativistic set up of the game in noninertial frames are different from the one obtained in Refs. [18, 19]. The creation operator (a_k) of particle and annihilation operator (b_k) of antiparticle in Minkowski space are related to the creation operator c_k^I in region I and annihilation operator $d_k^{II\dagger}$ in region II by the following Bogoliubov transformation

$$\begin{pmatrix} a_k \\ b_k^\dagger \end{pmatrix} = \begin{pmatrix} \cos r & -e^{-i\phi} \sin r \\ e^{i\phi} \sin r & \cos r \end{pmatrix} \begin{pmatrix} c_k^I \\ d_k^{II\dagger} \end{pmatrix}, \quad (4)$$

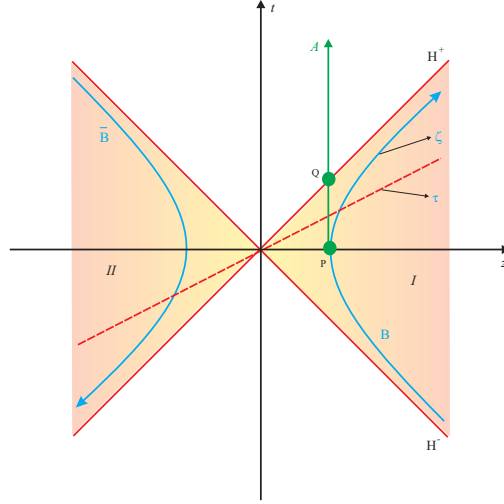


Figure 1: (color online) Rindler spacetime diagram: A uniformly accelerated observer Bob (B) moves on a hyperbola with constant acceleration a in region I and a fictitious observer anti-Bob (\bar{B}) moves on a corresponding hyperbola in causally disconnected region II . The coordinates τ and ζ are the Rindler coordinates in Bob's frame, which represent constant proper time and constant position, respectively. Lines H^\pm are the horizons that represent Bob's future and past and correspond to $\tau = +\infty$ and $\tau = -\infty$. Alice and Bob share an entangled initial state at point P and Q is the point where Alice crosses Bob's future horizon.

where k represents a single mode in each region and ϕ is an unimportant phase that can always be absorbed into the definition of the operators and r is the dimensionless acceleration parameter given by $\cos r = (e^{-2\pi\omega c/a} + 1)^{-1/2}$. The constants ω , c and a , in the exponential stand, respectively, for Dirac particle's frequency, speed of light in vacuum and Bob's acceleration. The parameter $r = 0$ when acceleration $a = 0$ and $r = \pi/4$ when $a = \infty$. We see that the transformation in Eq. (4) mixes a particle in region I and an antiparticle in region II . A similar transformation exists for an antiparticle's operator in region I and a particle's operator in region II [12]. In fact, a given Minkowski mode of a particular frequency spreads over all positive Rindler frequencies ($\omega/(a/c)$) that peaks about the Minkowski frequency [21, 22]. However, to simplify our problem we consider a single mode only in the Rindler region I , an

approximation that results into Eq. (4). This is valid if the observers' detectors are highly monochromatic that detects the frequency $\omega_A \sim \omega_B = \omega$.

From Eq. (4) one can find that

$$a_k = \cos r c_k^I - e^{-i\phi} \sin r d_k^{II\dagger}. \quad (5)$$

From the accelerated Bob's frame, with the help of Eq. (5), one can show that the Minkowski vacuum state is found to be a two-mode squeezed state

$$|0\rangle_M = \cos r |0\rangle_I |0\rangle_{II} + \sin r |1\rangle_I |1\rangle_{II}. \quad (6)$$

Note that in Eq. (6) we put I and II in the subscript of the kets to represent the Rindler modes in region I and region II , respectively. Eq. (6) shows that the noninertial observer that moves with a constant acceleration in region I sees a thermal state instead of the vacuum state. This effect is called the Unruh effect [23, 24]. Similarly, using the adjoint of Eq. (5) one can easily show that the excited state in Minkowski spacetime is related to Rindler modes as follow

$$|1\rangle_M = |1\rangle_I |0\rangle_{II}. \quad (7)$$

In terms of Minkowski mode for Alice and Rindler modes for Bob, the entangled initial state of Eq. (3) by using Eqs. (6) and (7) becomes

$$\begin{aligned} |\psi\rangle_{A,I,II} = & \cos \frac{\gamma}{2} \cos r |0\rangle_A |0\rangle_I |0\rangle_{II} \\ & + \cos \frac{\gamma}{2} \sin r |0\rangle_A |1\rangle_I |1\rangle_{II} + i \sin \frac{\gamma}{2} |1\rangle_A |1\rangle_I |0\rangle_{II}. \end{aligned} \quad (8)$$

Since Bob is causally disconnected from region II , we must take trace over all the modes in region II . This leaves the following mixed density matrix between the two players,

$$\rho_{A,BI} = \begin{pmatrix} \cos^2 r \cos^2 \frac{\gamma}{2} & 0 & 0 & -i \cos r \cos \frac{\gamma}{2} \sin \frac{\gamma}{2} \\ 0 & \cos^2 \frac{\gamma}{2} \sin^2 r & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i \cos r \cos \frac{\gamma}{2} \sin \frac{\gamma}{2} & 0 & 0 & \sin^2 \frac{\gamma}{2} \end{pmatrix}. \quad (9)$$

In the quantum Prisoners' Dilemma, the strategic moves of Alice and Bob are unitary operators which are given by [2]

$$\hat{U}_N(\alpha, \theta) = \begin{pmatrix} e^{i\alpha_N} \cos \frac{\theta_N}{2} & i \sin \frac{\theta_N}{2} \\ i \sin \frac{\theta_N}{2} & e^{-i\alpha_N} \cos \frac{\theta_N}{2} \end{pmatrix}, \quad (10)$$

where, the subscript $N = A, B$ represent Alice and Bob, $\theta \in [0, \pi]$ and $\alpha \in [0, 2\pi]$. If cooperation and defection are associated with the state $|0\rangle$ and the state $|1\rangle$, respectively, then the quantum strategy \hat{C} corresponds to $\hat{U}_N(0, 0)$ and the quantum strategy \hat{D} corresponds to $\hat{U}_N(0, \pi)$. To ensure that the classical game be a subset of the quantum one, Eisert et al. [2] argued that the operator \hat{J} must commute with the tensor product of any pair of the moves \hat{C} and \hat{D} .

Since fermionic system in noninertial frames is a physically realizable system, we hope that the encoding of the game might be practically possible. Once decisions are made, the final density matrix prior to the measurement becomes [2]

$$\rho = \hat{J}^\dagger \left(\hat{U}_A \otimes \hat{U}_B \right) \rho_{A,I} \left(\hat{U}_A^\dagger \otimes \hat{U}_B^\dagger \right) \hat{J}, \quad (11)$$

where \hat{J}^\dagger is applied to disentangle the final density matrix. The expected payoffs of the players are then found by using the following equation

$$P_N^{j_1 j_2} = \sum_i \$N^{j_1(i) j_2(i)} \rho_{ii}, \quad (12)$$

where ρ_{ii} ($i \in [0, 1]$) are the diagonal elements of the final density matrix and $\$N^{j_1(i) j_2(i)}$ ($j_1, j_2 \in [C, D]$) are the classical payoffs of the players from Table 1.

4 Results and discussion

The payoffs of the players for unentangled initial state ($\gamma = 0$), when each of them is allowed to play one of the two classical strategies, that is, $\hat{C} = \hat{U}_N(0, 0)$ or $\hat{D} = \hat{U}_N(0, \pi)$, are given in Table 2. The payoffs become the function of r .

Table 2: The payoff matrix of the players' payoffs as a function of the acceleration of Bob's frame. The first entry in every pair corresponds to Alice's payoff and the second entry corresponds to Bob's payoff. The initial state of the game is unentangled and the players are allowed to select a move from the two pure classical moves.

| | Bob: \hat{C} | Bob: \hat{D} |
|------------------|---------------------------|---------------------------|
| Alice: \hat{C} | $3 \cos^2 r, 4 - \cos 2r$ | $3 \sin^2 r, 4 + \cos 2r$ |
| Alice: \hat{D} | $3 + 2 \cos 2r, \sin^2 r$ | $3 - 2 \cos 2r, \cos^2 r$ |

One can easily see that the results of Table 2 reduce to the classical results of Table 1 when the acceleration $a = 0$ ($r = 0$). The presence of acceleration in the payoff functions of the players disturbs the symmetry of the game. Neither the strategy profile (\hat{C}, \hat{C}) nor the strategy profile (\hat{D}, \hat{D}) is an equilibrium outcome of the game in the range of acceleration $0 < r \leq \pi/4$. In this range of acceleration, Alice always wins by playing \hat{D} and always loses by playing \hat{C} . The dilemma like situation is resolved in the favor of Alice. At infinite acceleration ($r = \pi/4$), the strategy profiles $(\hat{C}, \hat{C}) = (\hat{C}, \hat{D}) = (3/2, 4)$, which means that if Alice plays \hat{C} , Bob strategy becomes irrelevant and he wins all the time. Similarly, the strategy profiles $(\hat{D}, \hat{C}) = (\hat{D}, \hat{D}) = (3, 3/2)$, Alice is victorious, regardless of what strategy Bob executes. Non of the strategy profiles is either Pareto optimal or Nash equilibrium.

However, for a maximal entangled state ($\gamma = \pi/2$), the situation is entirely different. When both the players are restricted only to the classical region of

moves, the payoffs of the players for different strategy profiles are given by

$$\begin{aligned}
P_{A,B}^{CC} &= 1 + \cos r + \cos^2 r + \frac{5}{4} \sin^2 r, \\
P_{A,B}^{DD} &= \frac{1}{8}(17 - 8 \cos r - \cos 2r), \\
P_A^{CD} &= P_B^{DC} = \frac{1}{2} \cos^2 \frac{r}{2} (9 + \cos r), \\
P_A^{DC} &= P_B^{CD} = \frac{1}{2} (9 - \cos r) \sin^2 \frac{r}{2}.
\end{aligned} \tag{13}$$

It can easily be seen from the payoffs function of Eq. (13) that the payoff

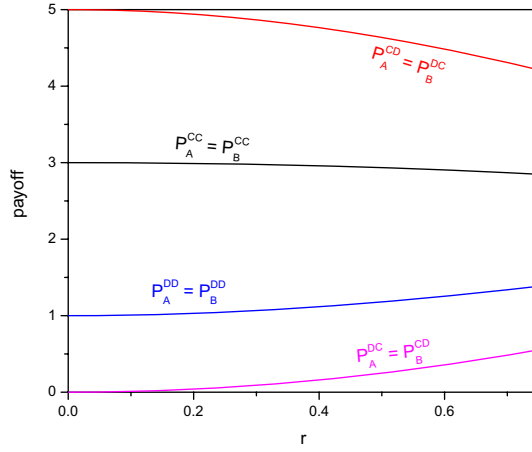


Figure 2: (color online) The payoffs for the maximally entangled initial state are plotted against the acceleration parameter r of Bob's frame. The players are allowed to choose only the classical moves. The subscripts stand for the players and the superscripts represent a strategy profile.

matrix is symmetric and that for $r = 0$, the classical results are obtained. Also, the strategy profiles (\hat{C}, \hat{C}) and (\hat{D}, \hat{D}) are equilibrium points for the whole range of the acceleration of Bob's frame. However, unlike the classical form and unentangled initial state of the quantum form in inertial frames of the game, the strategy \hat{C} in this case becomes the dominant strategy and it always results in payoff > 2.83 for all values of the acceleration of Bob's frame. Moreover, the

strategy profile (\hat{C}, \hat{C}) becomes the Nash equilibrium and the strategy profile (\hat{D}, \hat{D}) becomes the Pareto optimal of the game for all values of acceleration a . The payoffs of Eq. (13), as function of r for all the possible strategy profiles, are plotted in Fig. 2. It can be seen from the figure that playing \hat{C} is the best option for any player and hence resolves the dilemma like situation.

Now we consider the case in which the players are allowed to choose any strategy from the allowed quantum mechanical strategic space. We first consider the quantum strategy \hat{Q} of Eisert *et al.* [2], which is given by

$$\hat{Q} = \hat{U}(0, \pi/2) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \quad (14)$$

The payoffs of the players when Alice chooses \hat{Q} are given by

$$P_{A,B}^{Q\theta_B} = \frac{1}{4}[9 - \cos r((\cos r \mp 5) \cos \theta_B + 2 \cos 2\alpha_B(\cos \theta_B + 1) \pm 5)], \quad (15)$$

where $\theta_B = 0$ or π gives strategy \hat{C} or strategy \hat{D} respectively. Now, if Bob plays \hat{C} , then $P_A^{QC} = P_B^{QC}$ is an equilibrium point of the game. If Bob plays \hat{D} then $P_B^{QD} = P_A^{CD} > P_B^{QC} > P_A^{QD}$ for all values of the acceleration of the Bob's frame. This means that the quantum strategy \hat{Q} does no better for Alice against any of the two classical strategies of Bob. In other words, \hat{D} is the dominant strategy for Bob against Alice strategy \hat{Q} . The same is true for Alice, if Bob plays the quantum strategy \hat{Q} . In fact the strategy profile (\hat{Q}, \hat{C}) or (\hat{C}, \hat{Q}) is a Pareto optimal outcome. However, if both players execute \hat{Q} , the payoffs $P_A^{QQ} = P_B^{QQ} = P_{A,B}^{CC}$ and hence the strategy profile (\hat{Q}, \hat{Q}) is the Nash equilibrium.

Finally we consider the unfair game and the effect of the *miracle move* of Eisert *et al.* [2]. That is, if one player is restricted to the classical strategic space, then, in the case of inertial frames, the quantum player outsmarts the classical player all the time if he or she plays the miracle move \hat{M} ,

$$\hat{M} = \hat{U}\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (16)$$

However, this is not true in the case of noninertial frames. Let Alice plays \hat{M} and Bob is restricted to the classical strategies, the payoffs of the players become

$$\begin{aligned} P_A^{M\theta_B} &= \frac{1}{4}(-3 \cos^2 r \sin \theta_B + \cos r(\sin \theta_B - 7) + 9), \\ P_B^{M\theta_B} &= \frac{1}{4}(7 \cos^2 r \sin \theta_B + \cos r(\sin \theta_B + 3) + 9). \end{aligned} \quad (17)$$

It can easily be checked that $P_A^{M\theta_B} < P_B^{M\theta_B}$ irrespective of what strategy Bob executes. This result is symmetric with respect to the interchange of the players. That is, if Alice is restricted to the classical strategies and Bob plays \hat{M} , then, the payoffs of the players in Eq. (17) interchange and θ_B is replaced with θ_A .

The quantum player should never go for playing the quantum miracle move of the inertial frames. The dominance of quantum player over the classical one ceases in the case of noninertial frames. However, the miracle move \hat{M} always results in a winning payoff against the quantum move \hat{Q} . Logically, putting $r = 0$ in Eq. (17) should give the results of quantum Prisoners' Dilemma in the inertial frames but this is not so. Eq. (17) gives inverted results when $r = 0$, that is, Alice's payoff becomes Bob's payoff of the inertial frame and vice versa. We have no explanation for this inconsistency.

5 Conclusion

We study the influence of Unruh effect on the payoffs function of the players in the quantum Prisoners' Dilemma. For unentangled initial state, the Unruh effect gives rise to an asymmetric payoff matrix in contrast to the payoff matrix for the classical form and quantum form in the inertial frames of the game. It is shown that for unentangled initial state, Alice wins all the time if she plays \hat{D} and loses if she plays \hat{C} . As a result non of the classical strategies profile is either Perato optimal or Nash equilibrium. We have shown that the Unruh effect limits the dominance of the quantum player. The classical moves \hat{C} or \hat{D} becomes dominant against the quantum moves depending on the initial state entanglement. It is shown that the miracle move \hat{M} of the inertial frames becomes the worst move that always results in loss against any classical move. Nevertheless, against the quantum move \hat{Q} , it always gives a winning payoff. It is shown that the dilemma like situation is resolved in favor of one or the other player or for both players depending on the degree of entanglement in the initial state of the game.

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Figures Captions

Figure 1. (color online) Rindler spacetime diagram: A uniformly accelerated observer Bob (B) moves on a hyperbola with constant acceleration a in region I and a fictitious observer anti-Bob (\bar{B}) moves on a corresponding hyperbola in causally disconnected region II . The coordinates τ and ζ are the Rindler coordinates in Bob's frame, which represent constant proper time and constant position, respectively. Lines H^\pm are the horizons that represent Bob's future and past and correspond to $\tau = +\infty$ and $\tau = -\infty$. Alice and Bob share an entangled initial state at point P and Q is the point where Alice crosses Bob's future horizon.

Figure 2. (color online) The payoffs for the maximally entangled initial state are plotted against the acceleration parameter r of Bob's frame. The players are allowed to choose only the classical moves. The subscripts stand for the players and the superscripts represent a strategy profile.